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CONTACT PROBLEM FOR AN ELASTIC HALF-PLANE AND A SEMI-INFINITE ELASTIC ROD ADHERING TO IT

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The problem which will be considered is as follows. An elastic semi-infinite rod having constant cross section F is glued (or welded) to the side of an elastic semi-infinite plate having thickness h (see Fig. 1). At an arbitrary distance b from the end-face of the rod a unit force is applied in the direction of the rod axis. The contact shear stress $\tau_0(x)$ and the normal stress $\sigma_0(x)$ in an arbitrary cross section of the rod are to be found, assuming that the rod is not subjected to any bending moments (normal contact stress is not taken into account). A similar problem for an infinite rod was solved in [1]. The case of a semi-infinite rod was considered in [2, 3]; in [2] an approximate solution was given, while in [3] an exact solution was obtained

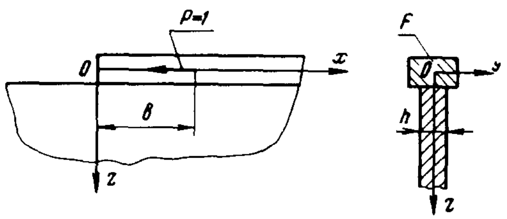

Fig. 1

(*). However, these papers dealt only with the simple case when the force is applied to the rod end-face.

We approach the problem in a radically different manner and present the exact solution for the case of a force applied at an arbitrary distance. The method is that suggested by one of the authors [4] for the solution of a similar problem of a force acting perpendicularly to the rod axis, with the shear contact stress completely neglected. The results of numerical computation of our exact solution are also given.

1. It is easily shown that the determination of the shear contact stress $\tau_0(x)$ and displacement of the rod $U_0(x)$ along axis x is equivalent to solving the following system of equations:

$$U_0(x) = -\frac{\gamma}{\pi}\int_0^\infty \ln\frac{1}{|x-t|}\,\tau_0(s)\,ds + \text{const} \qquad (-\infty < x < \infty) \tag{1.1}$$

$$U_0''(x) = -\alpha\,[\tau_0(x) - h^{-1}\delta(x-b)] \qquad (x \geqslant 0,\ \gamma = 2/E_0,\ \alpha = h\,(EF)^{-1})$$

with subsequent fulfilment of the boundary condition

$$U_0'(0) = 0 \tag{1.2}$$

Here $\delta(x)$ is the Dirac impulse function, E and E_0 are the elasticity moduli of the materials used for the rod and plate, respectively, and the prime denotes the derivative.

In the case of a unit force applied to the rod end ($b = 0$) it is more convenient to satisfy the following condition:

$$h\int_0^\infty \tau_0(x)\,dx = 1 \tag{1.3}$$

In order to solve equations (1.1) let us first consider an ancillary system

$$U_\lambda(x) = -\frac{\gamma}{\pi}\int_0^\infty K_0(\lambda|x-s|)\,\tau_\lambda(s)\,ds \qquad (-\infty < x < \infty)$$

$$U_\lambda''(x) - \lambda^2 U_\lambda(x) = -\alpha\,[\tau_\lambda(x) - h^{-1}\delta(x-b)] \qquad (x \geqslant 0) \tag{1.4}$$

Here $K_0(x)$ is the Macdonald function.

When Eqs. (1.4) are taken to the limit $\lambda \to 0$, we obtain Eqs. (1.1) because it is known that

$$K_0 z = O(\ln z), \qquad z \to 0$$

That the first equation of (1.4) passes to the first equation of (1.1) when $\lambda \to 0$ is also made clear in [5].

Thus, to solve system (1.1) it is sufficient to find $\tau_\lambda(x)$ from system (1.4), then take the result to the limit $\lambda \to 0$ and satisfy boundary condition (1.2).

2. Assuming $\lambda > 0$, direct substitution shows that the general solution of differential equations (1.4), vanishing at infinity, is given by the following function:

$$U_\lambda(x) = -C_\lambda e^{-\lambda x} + \lambda\int_0^\infty g_\lambda(x-s)\,[\tau_\lambda(s) - h^{-1}\delta(s-b)]\,ds \tag{2.1}$$

Here C_λ is an arbitrary constant and

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*) An exact solution was also given by Kalandiia [11] in a work published after the present paper was submitted to the Editor.

$$g_\lambda(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\eta x}}{\eta^2+\lambda^2}\,d\eta = \frac{e^{-\lambda|x|}}{2\lambda} \tag{2.2}$$

Equating (2.1) to the first equation of (1.4) yields the Wiener-Hopf integral equation of the first kind

$$\int_0^\infty l_\lambda(x-s)\,\tau_\lambda(s)\,ds = \frac{C_\lambda}{\gamma}e^{-\lambda x} + \frac{c}{h}g_\lambda(x-b) \qquad (x \geqslant 0)$$

$$l_\lambda(x) = cg_\lambda(x) + \pi^{-1}K_0(\lambda x) \qquad (c = \alpha/\gamma = hE_0(2EF)^{-1}) \tag{2.3}$$

which admits an exact solution.

It is more convenient to find first the solution of the equation

$$\int_0^\infty l_\lambda(x-s)\,\varphi_\zeta(s)\,ds = e^{i\zeta x} \qquad (x \geqslant 0, \operatorname{Im}\zeta \geqslant 0) \tag{2.4}$$

and then, allowing for the linearity of Eq. (2.3), its solution is obtained in the following form [5, 6, 4]:

$$\tau_\lambda(x) = \frac{C_\lambda}{\gamma}[\varphi_\zeta(x)]_{\zeta=i\lambda} + \frac{c}{2\pi h}\int_{-\infty}^{\infty} G_\lambda(-\zeta)\,\varphi_\zeta(x)\,d\zeta \tag{2.5}$$

where $G_\lambda(u)$ is the Fourier transform of function $g_\lambda(x-b)$.

The solution of Eq. (2.4) is derived by means of Hopf-Fock formula [7, 5, 6, 4].

$$\varphi_\zeta(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{\Psi(\omega)\,\Psi(\zeta)}{\omega+\zeta}\,e^{-ixw}\,dw \tag{2.6}$$

Here, function $\Psi(\omega)$ is regular and different from zero in the upper half-plane (excluding point ∞); it satisfies the functional equation

$$L^{-1}(\omega) = \Psi(\omega)\Psi(-\omega)$$

and its behavior at infinity is

$$\Psi(w) = O(\omega^\nu) \qquad (\omega \to \infty, \quad \nu < 1)$$

Function $L(\omega)$ is the Fourier transform of function $l_\lambda(x)$, the calculation of which results in the following formula:

$$\frac{1}{L(\omega)} = \frac{w^2+\lambda^2}{c+\sqrt{w^2+\lambda^2}} = H(w)$$

Thus, function $H(\omega)$ must be factorized. Removing from $H(\omega)$ factor $\sqrt{\omega^2+\lambda^2}$ which permits a self-evident factorization, we find in accordance with the general theory [7]

$$\Psi(w) = \sqrt{\lambda - iw}\,\chi_\lambda(w/\lambda) \tag{2.7}$$

where

$$\chi_\lambda(z) = \exp\left[-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \ln\left(1+\frac{c}{\lambda\sqrt{t^2+1}}\right)\frac{dt}{t-z}\right]$$

As shown in [8] (cf. also [6, 4, 9]), the last integral can be reduced to the following form:

$$\chi_\lambda(z) = \chi_\lambda(i\cos\tau) = \left(\frac{\cos\tau+1}{\cos\tau+\cos\sigma}\right)^{1/2}\exp\left[\frac{1}{2\pi}\int_{\tau-\sigma}^{\tau+\sigma}\frac{u}{\sin u}\,du\right], \quad \sin\sigma = -\frac{c}{\lambda} \tag{2.8}$$

By means of deformation of the integration path (this operation is described in greater detail in [9, 6]) into a loop embracing the ray $(-i\lambda, -i\infty)$ expression (2.6) can be written out as

$$\varphi_\zeta(x) = \frac{\Psi(\zeta)}{\pi}\int_\lambda^\infty \frac{R_\lambda(s)\,(s^2-\lambda^2)}{\Psi(is)\,(s+i\zeta)}\,e^{-xs}\,ds + H(\zeta)\,e^{i\zeta x} \tag{2.9}$$

$$R_\lambda(s) = \sqrt{s^2-\lambda^2}\,/\,(c^2+s^2-\lambda^2) \qquad (2.10)$$

Let us substitute expression (2.9) for $\varphi_\zeta(x)$ into (2.5) and change the order of integration in the iterated integral. Having done all this, let us perform the same operations, already carried out when deriving (2.9), on the integrals along infinite straight line. We have then instead of (2.5)

$$\tau_\lambda(x) = A\int_{\lambda/c}^{\infty} \frac{R_\lambda(cs)(cs+\lambda)}{\Psi(ics)}\, e^{-cxs}\, ds + T_\lambda(x) \qquad (2.11)$$

where

$$T_\lambda(x) = \frac{c^2}{h\pi^2}\int_{\lambda/c}^{\infty} \frac{R_\lambda(cs)(c^2s^2-\lambda^2)}{\Psi(ics)} J_\lambda(cs)\, e^{-cxs}\, ds + \frac{c^2}{h\pi}\int_{\lambda/c}^{\infty} R_\lambda(cs)\, e^{-c(x-b)s} ds \qquad (2.12)$$

$$J_\lambda(cs) = \int_{\lambda/c}^{\infty} \frac{R_\lambda(c\tau)\, e^{-cb\tau}}{\Psi(ic\tau)(\tau+s)}\, d\tau, \qquad A = \frac{cC_\lambda \Psi(i\lambda)}{\pi\gamma} \qquad (2.13)$$

If the force is applied to the end-face ($b = 0$), the second term on the right-hand side of Eq. (2.3) disappears and, consequently, similar terms in (2.5) and (2.11) disappear as well. Thus, in this case as well, the solution of system (1.4), allowing for (2.7), becomes

$$\tau_\lambda(x) = A\int_{\lambda/c}^{\infty} \frac{R_\lambda(cs)(cs+\lambda)}{\sqrt{\lambda+cs}\,\chi_\lambda(ics/\lambda)}\, e^{-cxs}\, ds \qquad (2.14)$$

3. As has been already said in the concluding sentence of Sect. 1, it is necessary to take to the limit $\lambda \to 0$ all expressions (2.11)–(2.13) if the solution of system (1.1) is to be found.

Let us first consider the case of a unit force applied to the end of the rod. In this case only formula (2.14) needs to be taken to the limit.

Let us note that $\displaystyle\lim_{\lambda\to 0} \chi_\lambda(ics/\lambda) = s^{1/2}(1+s^2)^{-1/4} \exp H_0(s)$ (3.1)

$$H_0(s) = \frac{1}{\pi}\left[\operatorname{arctg} s \ln s - \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^2}\, s^{2k+1}\right] \qquad (0 \leqslant s \leqslant 1)$$

The fact is here taken into account that function $\chi_\lambda(z)$ is the same as function $\chi_3(z;\lambda)$ defined by formulas (1.22) or (3.1) in [4], and that in [4] function $\chi_3(z;\lambda)$ is taken to the limit $\lambda \to 0$ and formula (3.1) is obtained in this manner. In the same paper (*) a very important computational feature of function $H_0(s)$ is revealed, namely that

$$H_0(s) = H_0(1/s) \quad (0 \leqslant s \leqslant 1) \qquad (3.2)$$

Allowing for (3.1) and (2.1), we obtain easily from (2.14)

$$\tau_0(x) = \lim_{\lambda\to 0} \tau_\lambda(x) = \frac{A^*}{c^{1/2}}\int_0^{\infty} \frac{s e^{-H_0(s)}}{(s^2+1)^{3/4}}\, e^{-cxs}\, ds \quad (A^* = \lim_{\lambda\to 0} A) \qquad (3.3)$$

which is the shear contact stress under the rod when the force is applied to the end of

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*) In this context let us draw attention to some misprints and errors in [4]. In formula (3.3) i in front of ∞ should be removed; in (3.4) it should be assumed that $c = 1$ and $\tau + \sigma_3 \to -\pi/2 - i\infty$; in (3.6) the integration limits are as follows: in the first integral $(-\pi/6 - i\infty, \pi/6 - i \ln s)$ and in the third integral $(0, -i\pi/6 - \ln s)$; in (3.8) the factor $(\dot{t} - i)^{-1/2}$ is left out; everywhere it should be taken that $(-\pi < \operatorname{Im}(\ln z) < \pi)$.

the rod.

With a view to deriving more convenient expressions both for shear contact stress $\tau_0(x)$ and normal stress in the rod $\sigma_0(x)$, we shall now introduce a dimensionless abscissa and normalized stresses $\tau(\xi)$ and $\sigma(\xi)$

$$\xi = cx,\ \beta = cb;\quad \tau(\xi) = h(2c)^{-1}\tau_0(\xi/c),\quad \sigma(\xi) = h(2c)^{-1}\sigma(\xi/c) \tag{3.4}$$

Allowing for (3.4) and (3.2), we have

$$\tau(\xi) = B\int_0^1 F(s)\left[se^{-\xi s} + \frac{e^{-\xi/s}}{s^{3/2}}\right]ds, \qquad B = \frac{A^*h}{2c^{3/2}} \tag{3.5}$$

$$\sigma(\xi) = 2\int_\xi^\infty \tau(\eta)\,d\eta = 2B\int_0^1 F(s)\left[e^{-\xi s} + \frac{e^{-\xi/s}}{s^{1/2}}\right]ds \tag{3.6}$$

$$F(s) = (s^2+1)^{-3/4}\,e^{-H_0(s)} \tag{3.7}$$

The arbitrary constant is found from condition (1.3)

$$B = \left[\int_0^1 F(s)\,(1+s^{-1/2})\,ds\right]^{-1} = 0.157$$

Formulas (3.5) and (3.6) were used for the computation of $\tau(\xi)$ (Table 1, $\beta = 0$) and $\sigma(\xi)$ (Table 2, $\beta = 0$). The results were virtually the same as in [2, 3].

4. Let us now consider a more general case, when a unit force is applied to an elastic rod at an arbitrary point $x = b > 0$. As has been already said, it is necessary to take formulas (2.11)–(2.13) to the limit to find $\tau_0(x)$, and afterwards to find the arbitrary constant from boundary condition (1.2).

Let us prove that this boundary condition will be satisfied if $A = 0$. To do this, it will be sufficient to show that the function

$$U_\lambda^*(x) = \int_0^\infty g_\lambda(x-s)\,f(s)\,ds$$

which for $f(x) = \alpha[\tau_\lambda(x) - h^{-1}\delta(x-b)]$ is a particular solution of the differential equation in system (1.4), has the following property:

$$\lim_{\lambda\to 0}\left[\frac{dU_\lambda^*(x)}{dx}\right]_{x=0} = 0 \tag{4.1}$$

Indeed, direct examination shows that

$$\left(\frac{d}{dx} - \lambda\right)g_\lambda(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\eta x}}{\eta - i\lambda}\,d\eta = 0 \qquad (x < 0)$$

The consequence of this is that

$$\left[\left(\frac{d}{dx} - \lambda\right)U_\lambda^*(x)\right]_{x=0} = 0$$

In this manner, (4.1) is now proved.

Thus $\tau_0(x)$ can be obtained if only formulas (2.12) and (2.13) are taken to the limit $\lambda \to 0$. Let us carry out on the integrals in (2.12) and (2.13) the same operations which were performed earlier when deriving (3.5) and (3.6). We have then

$$\tau(\xi) = \frac{1}{2\pi^2}\int_0^1 F(s)\left[sJ(s)\,e^{-\xi s} + J\left(\frac{1}{s}\right)\frac{e^{-\xi/s}}{s^{3/2}}\right]ds + \tau_\infty(\xi - \beta) \tag{4.2}$$

$$\sigma(\xi) = \frac{1}{\pi^2}\int_0^1 F(s)\left[J(s)\,e^{-\xi s} + J(s^{-1})\,\frac{e^{-\xi/s}}{s^{1/2}}\right]ds + \sigma_\infty(\xi - \beta) \tag{4.3}$$

where

$$J(z) = \int_0^1 F(t)\left[\frac{e^{-\beta t}}{1 + t/z} + \frac{t^{1/2}\,e^{-\beta/t}}{t + z^{-1}}\right]dt$$

$$\tau_\infty(z) = \frac{1}{2\pi}\int_0^\infty \frac{te^{-zt}}{t^2+1}\,dt = \frac{1}{2\pi}(-\operatorname{ciz}\cos z - \operatorname{siz}\sin z) \quad (z > 0) \tag{4.4}$$

$$\sigma_\infty(z) = \frac{1}{\pi}\int_0^\infty \frac{e^{-zt}}{t^2+1}\,dt = \frac{1}{\pi}(\operatorname{ciz}\sin z - \operatorname{siz}\cos z) \quad (z > 0) \tag{4.5}$$

Function $F(s)$ is determined by expression (3.7).

When calculating the integrals in (4.4) and (4.5), use was made of formulas given in [10] (p. 326). These expressions determine normalized stresses in the case of an infinite rod loaded by a unit force at $z = 0$, and completely agree with formulas in [1] obtained by a different method.

Using (4.2)–(4.5), the values of $\tau(\xi)$ (Table 1) and $\sigma(\xi)$ (Table 2) were computed as functions of normalized distance ($\beta = cb$) between the point where the unit force is applied and the end of the rod.

Table 1

β	$\xi = 0.2$	0.4	0.6	1.0	1.4	2.0
0.0	0.429	0.236	0.157	0.087	0.056	0.033
0.2	∞	0.289	0.180	0.095	0.060	0.035
0.4	0.343	∞	0.249	0.113	0.068	0.038
0.6	0.243	0.266	∞	0.147	0.081	0.043
1.0	0.155	0.138	0.158	∞	0.138	0.060
1.4	0.119	0.093	0.094	0.141	∞	0.096

Table 2

β	$\xi = 0.2$	0.4	0.6	1.0	1.4	2.0
0.0	0.564	0.438	0.361	0.268	0.212	0.161
0.2	—0.347*	0.474	0.384	0.280	0.220	0.165
0.4	—0.227	—0.404*	0.436	0.304	0.234	0.173
0.6	—0.180	—0.278	—0.435*	0.341	0.254	0.184
1.0	—0.129	—0.186	—0.244	—0.464*	0.324	0.216
1.4	—0.100	—0.141	—0.178	—0.267	—0.478*	0.270

These numerical values clearly indicate that for $\beta \geqslant 1.4$ the calculations may be carried out using formulas for an infinite rod.

When Table 2 is used, it must be borne in mind that values with asterisks are valid for the cross section on the left-hand side of the point at which the unit force is applied. For the values valid for the cross section to the right of the force application point, unity must be added to the values in the Table.

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A METHOD FOR THE CONSTRUCTION OF THE APPROXIMATE SOLUTION OF THE MIXED AXISYMMETRIC PROBLEM IN THE THEORY OF ELASTICITY

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In this work some general considerations are presented with respect to the construction of an approximate solution to spatial mixed problems in the theory of elasticity. The axisymmetric problem is used as an example.

For the solution a structure is proposed which permits to satisfy exactly mixed boundary conditions of a certain type. In addition, this structure contains a series of arbitrary functions the selection of which can be made such that the system of differential equations for the equilibrium of the elastic body is satisfied in the best possible manner (in one sense or another).

The analyses are based on the utilization of R-functions [1] which makes it possible to examine practically any real three-dimensional bodies. The question of the foundation of the method is not discussed.